f1a)

i)

null(A) = (0, 0)

range(A) = 

null(AT ) = span 

range (AT ) = 

ii)

Null(AT): dimensional line

Range(A): dimensional plane orthogonal to each other

Null(A): single point at origin

Range(AT): all of R2 orthogonal to each other.

b) i)

ii) √4 = 2 and √2. ||A||2=√4 = 2

iii)

AT= (USVT)T

AT= V(US)T

AT= VSTUT is the SVD decomposition of AT.

iv)

V is the new basis of R^2 and U is the new basis of R^3

And the new transformation is S.

c) i)

Important to note that rank(B) = rank(BT)

First show rank(B) = rank(BTB):

Need to show null(B) = null(BTB) to show rank(B) = rank(BTB):

Bx = 0

=> BTBx = 0

=> If x is part of null(B) then x is part of null(BTB)

BTBx = 0

=> xTBTBx = 0

=> (Bx)TBx = 0

=> Bx = 0

=> If x is part of null(BTB) then x is part of null(B)

So null(B) = null(BTB)

=> nullity(B) = nullity(BTB)

=> n - rank(B) = n - rank(BTB) (by Rank Nullity Theorem)

=> rank(B) = rank(BTB)

Now let’s say we want to calculate rank(BT). By the above:

rank(BT) = rank((BT)T(BT)) (by substitution into rank(A) = rank(ATA) with A = BT)

= rank(BBT)

Because rank(B) = rank(BT), we have rank(B) = rank(BTB) = rank(BT) = rank(BBT)

ii) For each eigenvalue μ of B, B+cI has eigenvalue μ+c associated with the same eigenvector.

Indeed we have Bu = μu for each eigenvector u and associated eigenvalue μ, thus:

So B and B+cI have the same set of eigenvectors. Hence, their spectral decompositions are similar: S’ = QT(B+ci)Q is the decomposition for B+cI. mnj

S’ is obtained by taking S and adding c to all non-zero elements.

Maybe a simpler way? Since QT = Q-1

B = QSQT

B + cI = QSQT + cI

QT(B + cI) = SQT + cQT

QT(B + cI)Q = S + cI

B+cI = Q(S+cI)QT

2a) perform the QR decomposition as describes in the slides gives us:

e\_1 = (½) \* [1,1,1,1] ^ T

e\_2 = (½) \* [1,-1,1,-1] ^ T

e\_3 = (½) \* [-1,-1,1,1] ^ T

e\_4 = (½) \* [-1,1,1,-1] ^ T

Thus Q = (1/2) \* [[1, 1,-1,-1],

[1,-1,-1, 1],

[1, 1, 1, 1],

[1,-1, 1,-1]]

Calculating R gives us:

R = [[2,1,1,3/2], [0,1,0,1/2], [0,0,1,-1/2],[0,0,0,1/2]]

Calculating Q^T \* [2,2,0,2]^T gives us [3,-1,-1,-1], and let this vector be **b**

Now we solve **Rx** = **b**

Using backward substitution gives us:

x\_4 = -2

x\_3 = -2

x\_2 = 0

x\_1 = 4

And by substituting it back we know it is indeed the case.

b) check Piazza

i)

l\_1 norm is 10

l\_infinitiy norm is 6

ii)

A sketch proof for the lower bound is that assuming the maximum elements in terms of absolute is a\_km and in vector x we set x\_m to be 1 and all other elements 0.

In other words:

For LHS

By def of our induced norm, we must maximise ||Au|| where ||u|| = 1. If one value of u is 1 and all others are 0, the norm ||Au|| > max entry of A by definition of an l-norm

(also proved ||Au|| infinity <= ||Au||2 in lectures). If u isn’t composed in such a way it must be the fact that ||Au|| would be greater (or equal) to if u were composed that way, so it still captures the max entry.

For RHS

To show

||A||^2 = (expanded definition of induced l-2 norm with matrix norm)^2 <= nm|max entry|^2

To explain this, we first assume a vector u of all 1/sqrt(m)s. This way, ||u|| = 1 and the norm is equally distributed upon the values of u. Now we find ||Au||^2 is upper bounded

by n \* |m\*max entry\* 1/sqrt(m)|^2. N due to the summation of the induced norm. M max entries \* 1/sqrt(m) because we have m columns added together via matrix multiplication. Upper bounded by the max entry because that is the highest value any entry could possibly be. This way we have our general upper bound for our induced norm.

Simple Algebra will give

||Au||^2 <= n \* |m \* max entry \* 1/sqrt(m)|^2 = n \* m \* |max entry|^2

=>

||Au|| <= sqrt(n \* m) \* |max entry|